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Lower bounds for blow up time of the p-Laplacian equation with damping term

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Abstract. In this work deals with the p-Laplacian wave equation with damping terms in a bounded domain. Under suitable conditions, we obtain a lower bounds for the blow up time. Our result extends the recent results obtained by Baghaei (2017) and Zhou (2015), for $p > 2$.

1. Introduction

In this work, we study the following p-Laplacian equation with strong and weak damping terms

(1)
$$\begin{cases} u_{tt} - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) - a\Delta u_t + bu_t = |u|^{q-2}u, & x\in\Omega, \quad t>0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x\in\Omega, \\ u(x,t) = 0, & x\in\partial\Omega, \ t>0, \end{cases}$$

where $\Omega \subset R^n$ $(n = 2, 3, \dots)$ is a bounded domain with a smooth boundary $\partial\Omega$; and $u_0(x) \in W_0^{1,p}(\Omega)$, $u_1(x) \in L^2(\Omega)$, $p > 2$. $a \geq 0$, $b > -a\rho_1$ with $\rho_1 > 0$ is the first eigenvalue of the operator$-\Delta$ under homogeneous Dirichlet boundary conditions and

$$\begin{cases} 2 < q < \infty \ \text{ if } n = 2, \\ 2 < q \leq \begin{cases} \frac{2n}{n-2}, & \text{for } a > 0 \\ \frac{2n-2}{n-2}, & \text{for } a = 0 \end{cases} \ \text{ if } n \geq 3. \end{cases} \tag{2}$$

When $p = 2$, (1) is reduced to the following wave equation

$$u_{tt} - \Delta u - a\Delta u_t + bu_t = |u|^{q-2}u. \tag{3}$$

In 2006, Gazzalo and Squassina [2] studied problem (3). They proved the local existence, global existence and blow up of solutions. Later, some authors studied the lower bounds for the blow up time under some conditions,

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see [1,7,9]. Also, in recent years, some authors investigate the lower bounds for blow up time for hyperbolic type equations, see [3,4,6,8].

Inspired by the above papers, in this paper we consider the lower bound for the blow up time of solutions (1). Our result improves the recent results obtained by Baghaei [1] and Zhou [9], for $p > 2$.

Now, for the problem (1), we define the functionals, the potential well depth d and the unstable set U, are given as

$$\begin{aligned} I(u) &= \|\nabla u\|_p^p - \|u\|_q^q, \\ J(u) &= \frac{1}{p}\|\nabla u\|_p^p - \frac{1}{q}\|u\|_q^q, \\ E(t) &:= E(u,u_t) = J(u) + \frac{1}{2}\|u_t\|^2, \\ d &= \inf_{u\in H_0^1(\Omega)/\{0\}} \max_{\lambda\ge 0} J(\lambda u), \\ U &= \left\{u \in W_0^{1,p} : J(u) \le d \text{ and } I(u) < 0\right\} \end{aligned} \tag{4}$$

with above assumptions, the results on the existence of local solutions and the nonexistence of the solutions of the problem (1) can be reformulated as follows:

(i) Assume that q satisfy (2), then there exist $T > 0$ and a unique solution u problem (1) satisfying

$$\begin{aligned} u &\in C\left([0,T), W_0^{1,p}(\Omega)\right) \cap C^1\left([0,T), L^2(\Omega)\right) \cap C^2\left([0,T), H^{-1}(\Omega)\right), \\ u_t &\in L^2\left((0,T), H_0^1(\Omega)\right). \end{aligned}$$

(ii) The solution u of problem (1) blows up at a finite time T if and only if there exists $\bar{t} \in [0,T)$ such that $u(\bar{t}) \in U$ and $E\left(u(\bar{t}), u_t(\bar{t})\right) \le d$.

$$\begin{aligned} E(t) &= \frac{1}{2}\|u_t\|^2 + \frac{1}{p}\|\nabla u\|_p^p - \frac{1}{q}\|u\|_q^q, \\ E(0) &= \frac{1}{2}\|u_1\|^2 + \frac{1}{p}\|\nabla u_0\|_p^p - \frac{1}{q}\|u_0\|_q^q. \end{aligned} \tag{5}$$

Lemma 1 ([5]). *Let $\Omega \subset R^n$ $(n \ge 2)$. Let u be a non-negative piecewise C^1 function defined in Ω with $u(x) = 0$, $x \in \partial\Omega$. So*

$$\int_\Omega |u|^{2s}\,\mathrm{d}\,s \le \delta\left(\int_\Omega |\nabla u|^2\,\mathrm{d}\,x\right)^s \tag{6}$$

satisfies for

$$\begin{cases} s > 1, & \text{if } n = 2, \\ 1 < s < \frac{n}{n-2}, & \text{if } n \ge 3, \end{cases}$$

with $\delta = \left(\frac{n-1}{n^{\frac{3}{2}}}\right)^{2s} |\Omega|^{1-\frac{(n-2)}{n}s}$.

Theorem 1. *Assume that $q > \frac{p+2}{2}$ holds. Let $u(x,t)$ be the solution of the problem (1), which blows up at a finite time T^*. Then*

$$\int_{\psi(0)}^{\infty} \frac{\mathrm{d}\tau}{qM + \tau + \varepsilon q p^{\frac{2(q-1)}{p}} 2^{\frac{2(q-1)}{p}-2} M^{\frac{2(q-1)}{p}} + \varepsilon p^{\frac{2(q-1)}{p}} q^{\frac{p-2q+2}{p}} 2^{\frac{2(q-1)}{p}-2} \tau^{\frac{2(q-1)}{p}}} \leq T^*,$$

where

$$\left\{ \begin{array}{l} \psi(0) = \|u_0\|_q^q, \\ \varepsilon = \left(\frac{n-1}{n^{\frac{3}{2}}}\right)^{2(q-1)} |\Omega|^{1-\frac{(n-2)(q-1)}{n}}, \\ M = \frac{1}{2}\|u_1\|^2 + \frac{1}{p}\|\nabla u_0\|_p^p - \frac{1}{q}\|u_0\|_q^q. \end{array} \right. \tag{7}$$

Proof. Multiplying the equation of (1) by u_t, then integrating the result over Ω, we have

$$\int_\Omega u_t \left[u_{tt} - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) - a\Delta u_t + b u_t\right] \mathrm{d}x = \int_\Omega u_t |u|^{q-2} u \,\mathrm{d}x,$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u_t|^2 \mathrm{d}x + \frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u|^p \mathrm{d}x + a\int_\Omega |\nabla u_t|^2 \mathrm{d}x + b\int_\Omega |u_t|^2 \mathrm{d}x = \frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u|^q \mathrm{d}x.$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|u_t\|^2 + \frac{1}{p}\|\nabla u\|_p^p - \frac{1}{q}\|u\|_q^q\right) = -a\|\nabla u_t\|^2 - b\|u_t\|^2,$$

the above calculations imply $E'(t) \leq 0$, with

$$E(t) := \frac{1}{2}\|u_t\|^2 + \frac{1}{p}\|\nabla u\|_p^p - \frac{1}{q}\|u\|_q^q.$$

So,

$$E(t) \leq E(0) = M, \tag{8}$$

where M is defined in (7) and

$$\|u_t\|^2 + \frac{2}{p}\|\nabla u\|_p^p = 2E(t) + \frac{2}{q}\|u\|_q^q \leq 2M + \frac{2}{q}\|u\|_q^q. \tag{9}$$

Now define

$$\psi(t) = \|u\|_q^q,$$

then thanks to Cauchy's and embedding ($L^p(\Omega) \hookrightarrow L^2(\Omega)$, $p > 2$) inequalities and Lemma 1 with $s = q-1$ and $\delta = \varepsilon$, we obtain

$$\begin{aligned} \psi'(t) &= q\int_\Omega |u|^{q-2} u u_t \,\mathrm{d}x \\ &\leq \frac{q}{2}\left(\|u_t\|^2 + \|u\|_{2(q-1)}^{2(q-1)}\right) \\ &\leq \frac{q}{2}\left(\|u_t\|^2 + \varepsilon\|\nabla u\|_2^{2(q-1)}\right) \end{aligned}$$

$$\begin{aligned} &\leq \frac{q}{2}\left(\|u_t\|^2+\varepsilon\|\nabla u\|_p^{2(q-1)}\right) \\ &= \frac{q}{2}\left(\|u_t\|^2+\varepsilon\left(\|\nabla u\|_p^p\right)^{\frac{2(q-1)}{p}}\right). \end{aligned}$$

By the (9) and definition of ψ, we get

$$\begin{aligned} \psi'(t) &\leq \frac{q}{2}\left[2M+\frac{2}{q}\psi(t)+\varepsilon\left(\frac{p}{2}\right)^{\frac{2(q-1)}{p}}\left(2M+\frac{2}{q}\psi(t)\right)^{\frac{2(q-1)}{p}}\right] \\ &\leq \frac{q}{2}\left[\begin{array}{c} 2M+\frac{2}{q}\psi(t) \\ +\varepsilon\left(\frac{p}{2}\right)^{\frac{2(q-1)}{p}}2^{\frac{2(q-1)}{p}-1}\left((2M)^{\frac{2(q-1)}{p}}+\left(\frac{2}{q}\psi(t)\right)^{\frac{2(q-1)}{p}}\right)\end{array}\right] \\ &= \frac{q}{2}\left[2M+\frac{2}{q}\psi(t)+\varepsilon\frac{p^{\frac{2(q-1)}{p}}}{2}\left((2M)^{\frac{2(q-1)}{p}}+\left(\frac{2}{q}\psi(t)\right)^{\frac{2(q-1)}{p}}\right)\right] \\ &= qM+\psi(t)+\varepsilon q\frac{p^{\frac{2(q-1)}{p}}}{4}\left(2^{\frac{2(q-1)}{p}}M^{\frac{2(q-1)}{p}}+\frac{2^{\frac{2(q-1)}{p}}}{q^{\frac{2(q-1)}{p}}}\psi^{\frac{2(q-1)}{p}}(t)\right) \\ &= qM+\psi(t)+\varepsilon qp^{\frac{2(q-1)}{p}}2^{\frac{2(q-1)}{p}-2}M^{\frac{2(q-1)}{p}} \\ &\quad +\varepsilon p^{\frac{2(q-1)}{p}}q^{\frac{p-2q+2}{p}}2^{\frac{2(q-1)}{p}-2}\psi^{\frac{2(q-1)}{p}}(t), \end{aligned}$$

where used $(a+b)^{\varpi}\leq 2^{\varpi-1}\left(a^{\varpi}+b^{\varpi}\right)$. Thus, the above inequality implies that

$$\begin{aligned} \frac{\mathrm{d}\,\psi(t)}{\mathrm{d}\,t} &\leq qM+\psi(t)+\varepsilon qp^{\frac{2(q-1)}{p}}2^{\frac{2(q-1)}{p}-2}M^{\frac{2(q-1)}{p}} \\ &\quad +\varepsilon p^{\frac{2(q-1)}{p}}q^{\frac{p-2q+2}{p}}2^{\frac{2(q-1)}{p}-2}\psi^{\frac{2(q-1)}{p}}(t). \end{aligned} \tag{10}$$

Since $\lim_{t\to T^*}\psi(t)=\infty$, we get from (10)

$$\int_{\psi(0)}^{\infty}\frac{\mathrm{d}\,\tau}{qM+\tau+\varepsilon qp^{\frac{2(q-1)}{p}}2^{\frac{2(q-1)}{p}-2}M^{\frac{2(q-1)}{p}}+\varepsilon p^{\frac{2(q-1)}{p}}q^{\frac{p-2q+2}{p}}2^{\frac{2(q-1)}{p}-2}\tau^{\frac{2(q-1)}{p}}}\leq T^*.$$

This completes the proof of the main theorem. □

2. CONCLUSION

In this work, we obtained the lower bounds for blow up time of the nonlinear p-Laplacian equation with damping terms in a bounded domain. This improves and extends many results in the literature.

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